## Multidefender Security Games

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#### Abstract

Stackelberg security game models and associated computational tools have seen deployment in a number of high-consequence security settings, such as LAX canine patrols and Federal Air Marshal Service. These models primarily focus on isolated systems where only one defender is present, despite being part of a more complex system with multiple players. Furthermore, many real systems such as transportation networks and the power grid exhibit interdependencies between targets and, consequently, between decision makers jointly charged with protecting them. In order to understand such multi-defender strategic interactions present in security, we investigate game theoretic models of security games with multiple defenders. Unlike most prior analysis, we specifically focus on the situations in which each defender must protect multiple targets, so that even a single defender's best response decision is, in general, highly non-trivial. We start with an analytical investigation of multi-defender security games with independent targets, offering an equilibrium and price-of-anarchy analysis of three models with increasing generality. In all models, we find that the defenders have the incentive to over-protect the targets, at times significantly. Additionally, in the simpler models, we find that the price of anarchy is unbounded, linearly increasing both in the number of defenders and the number of targets per defender. Considering interdependencies among targets, we develop a novel mixedinteger linear programming formulation to compute a defender's best response, and make use of this formulation in approximating Nash equilibria of the game. We apply this approach towards computational strategic analysis of several models of networks representing interdependencies, including real-world power networks. Our analysis shows how network structure and the probability of failure spread determine the propensity of defenders to over- or under-invest in security.

#### 1. Introduction

Security, physical and cyber, has come to the forefront of national attention, particularly after 9/11. Among the variety of approaches that are used to tackle security problems, from risk analysis to red teaming, game theory has had a significant impact, with tools based on game theoretic analysis having been deployed in LAX airport to schedule canine patrols (Paruchuri et al., 2008; Jain et al., 2008; Pita et al., 2009), by Federal Air Marshall Service (FAMS) to schedule the air marshals (Kiekintveld et al., 2009; Jain et al., 2010, 2010), and by the US Coast Guard to schedule boat patrols (Shieh et al., 2012). All of these deployments, and numerous other related efforts, have cast security as a Stackelberg game between a single defender and an attacker, in which the defender leads (i.e., acts first),

choosing a probability distribution over defense actions, and the attacker, upon learning this probability distribution, chooses a response (Conitzer & Sandholm, 2006). In many cases, the attacker is modeled as a rational agent who selects an optimal response and, in the many applications that compute a Strong Stackelberg equilibrium, an attacker is often assumed to break ties in the defender's favor (Paruchuri et al., 2008; Korzhyk et al., 2011).

A crucial assumption that all these efforts have in common is that a single defender is responsible for all the targets that need protection, and that she has control over all of the security resources. However, there are many domains in which there are multiple defender agencies who are in charge of different subsets of all targets. In practice, numerous parties are responsible for security; indeed, the fact that the basic framework has been deployed by different entities and agencies makes this manifest already. If security decisions made by different parties were entirely independent, both from the defender's and the attacker's perspective, a single-defender model would be entirely satisfactory. However, the assets protected by different entities are typically interdependent, or, more generally, have value to others who are not involved in security decisions. Additionally, attackers, insofar as they may target different sectors under the charge of different defenders, are resource constrained, implicitly coupling otherwise independent targets.

An important motivating application for our multidefender security game is security and reliability in the power grid. Independent System Operators (ISOs) and profit-driven independent utility operators are largely responsible for operating and controlling subsystems of the entire grid (Lazar, 2011). These operators are held responsible for the reliability of their system, and thus have independent, and possibly even competing, goals with neighboring ISOs. As such, their security decisions are made independently, despite the interdependencies present between subsystems. As a result of this organization, cascading failures in the power grid can present a great threat to the entire system, even when an explicit attacker is not present. This problem is exacerbated by the fact that components in the grid are controlled by multiple entities and are also dependent on other independently operated utility networks (water, communications, natural gas, etc.).

We extend the previous Stackelberg game models in two ways:

- 1. an analytic equilibrium and price of anarchy (PoA) characterization of 3 multidefender security scenarios, in which we assume *homogeneous and independent valuations* of the targets for each defender; and
- 2. a computational analysis leveraging a novel mixed-integer linear program (MIP) approach for computing a defender's best response, combined with a novel heuristic method for approximating equilibria in interdependent multi-defender security games with heterogeneous targets.

In case 1 where there are multiple defenders, and the values of the targets are *independent* and *homogeneous* among the defenders, our analysis is focused on three models of such multi-defender games (each varying in their level of generality). We show that a Nash equilibrium among defenders in this two-stage game model need not always exist, even when the defenders utilize randomized strategies (i.e., probability distributions over target protection levels); this is distinct from a model in which the attacker moves simultaneously with the defenders, where a mixed strategy equilibrium is guaranteed to exist. When

an equilibrium does exist, we show that the defenders protect all of their targets with probability 1 in all three models, whereas the socially optimal protection levels are generally significantly lower. When no equilibrium exists, we characterize the best approximate Nash equilibrium (that is, one in which defenders have the least gain from deviation), showing that over-investment is substantial in this case as well. Our *price of anarchy* (PoA) analysis, which relies on the unique equilibrium when it exists, and the approximate equilibrium otherwise, demonstrates a surprising finding: whereas PoA is unbounded in the simpler models, increasing linearly with the number of defenders, the more general model shows this to be an atypical special case achieved when several parameters are exactly zero. More generally, PoA is bounded by a constant.

For case 2, we introduce interdependencies between targets. Because closed-form analysis in this setting is intractable, we propose a novel mixed-integer linear programming approach combined with a novel heuristic method to approximate equilibrium behavior. Unlike other multi-defender models (e.g., (Kunreuther & Heal, 2003; Chan et al., 2012; Bachrach et al., 2013; Acemoglu et al., 2013)), our approach maintains the typical complexity of individual defender decision process in the multi-defender framework, with each defender responsible for securing many, possibly interdependent, targets. Our setup gives rise to two competing externalities of security decisions: a positive externality, where greater security implies reduced contagion risk to other defenders, and a negative externality, which arises because high security by one player pushes the attacker to attack someone else's assets. We study the impact of competing externality effects of defense on the resulting Nash equilibrium outcomes as a function of network topology (using both synthetic and real networks), interdependent risk, and the level of system decentralization. One of our key findings is that the impact of system decentralization on security and welfare can be non-monotonic when systems are highly interdependent: high levels of decentralization can yield near-optimal outcomes, even as moderate decentralization results in significant underinvestment. With weak interdependencies, on the other hand, an increasingly decentralized system tends more strongly to over-invest in security.

The remainder of our paper is outlined as follows. In Section 2, we give an overview of related work. In Section 3, we briefly outline the definitions and solution concepts of our independent and interdependent multi-defender security games, respectively. Section 4 provides an equilibrium and PoA analysis of the homogeneous, independent security game models. Section 5 further explains the interdependent multi-defender model, and presents results on well-studied synthetic networks, as well as on real-world power grid networks.

#### 2. Related Work

Our work, like much work in the recent security game literature, builds on the notion of Stackelberg games (Osborne & Rubenstein, 1994), which model commitment in strategic settings. The first thorough computational treatment of randomized (mixed strategy) commitment was due to Conitzer and Sandholm (2006). In this line of work, of greatest relevance to our effort are multiple-leader Stackelberg games (Sherali, 1984; DeMiguel & Xu, 2009; Leyffer & Munson, 2007; Rodoplu & Raj, 2010; Kulkarni & Shanbhag, 2014; Sinha et al., 2014). In many cases, these approaches leverage specialized problem structure, and are not immediately applicable to our setting. In particular, Sherali (1984) and DeMiguel

and Xu (2009) focus on relatively simple models with firms setting production quantity (a single variable), aiming to maximize profit. Both show existence and uniqueness of equilibria in their setting, and leverage these characterization results to obtain solutions to the games. Similarly, Rodoplu and Raj (2010) consider a relatively simple model of network competition in which leaders are nodes setting prices for packets transmitted through them; again, each leader only sets a single variable, the utility functions are problem-specific, and algorithms are specialized to the particular problem structure (and are inapplicable to our setting).

Sinha et al. (2014) propose an evolutionary algorithm for solving bi-level Stackelberg problems, but their problem structure is also highly specific to the domain of interest (firms choosing production, investment, and marketing, and maximizing profit), and the evolutionary algorithm leverages significant simplifications, such as the assumption that the market eventually clears. Leyffer and Munson (2007) present a very generic multi-leader multi-follower setting and solution framework in the context of shared complementarity constraints (which is the case for our problem, where a single follower attacks a single target), but rely on separability of objective functions in leader and follower variables, the assumption that does not obtain in our setting (in addition, their approach only scales to 2-4 leaders, whereas we are able to approximately solve games with 64 leaders). Kulkarni and Shanbhag (2014) offer a deep theoretical treatment of a relatively broad class of multi-leader multi-follower games, but much of their analysis and positive results are restricted to potential games, and they do not offer specific algorithmic suggestions. Like us, they leverage shared constraints to resolve the issue of incompatible leader assumptions about the follower tie-breaking behavior.

Our point of departure is a class of Stackelberg games specifically pertinent to security: commonly, these are simply known as *security games* (Korzhyk et al., 2011; Paruchuri et al., 2008; Jain et al., 2010; Vorobeychik et al., 2011). In these games, a single defender allocates a set of resources among potential targets of attack in a randomized fashion (that is, the defender commits to a probability distribution over resource-to-target mappings), with an attacker choosing a single target to attack after observing the defender's strategy. Almost universally in this domain, a Strong Stackelberg equilibrium (SSE) is a solution concept of choice. In SSE, the follower (attacker) is assumed to break ties in the defender's favor. As we will see below, this solution concept presents conceptual and technical problems in a multi-defender setting.

A similar, though mostly orthogonal, line of work are network interdiction problems (Cormican et al., 1998; Woodruff, 2003), in which a leader attempts to interdict a network over which the follower subsequently solves a variation of a network flow problem. Unlike the literature on security games, as well as our setting, network interdiction problems are almost universally zero-sum (minimax).

Another somewhat related line of work considers the problem of coordination and teamwork among multiple defenders in a purely cooperative setting (Jiang, Procaccia, Qian, Shah, & Tambe, 2013; Shieh, Jiang, Yadav, Varakantham, & Tambe, 2014a). This work, however, is entirely unlike ours: in particular, our primary focus is on the impact of incentive misalignment among the defenders with different (though certainly related) motivations, rather than coordination issues and teamwork. While often effective coordination among multiple defenders can be achieved, just as often (if not predominantly) decentral-

ization of decision making processes and resources inherently give rise to distinct, and often conflicting, incentives among defenders.

Among the earliest multi-defender models is the literature on interdependent security games (Kunreuther & Heal, 2003), in which interactions among multiple defenders are modeled as an n-player, 2-action game, where a player decides whether to invest in security; however, no attacker is considered. More recently, time-dependent scenarios where coordination of defender resources amongst multiple defenders is assumed have been studied using Markov decision processes (Shieh et al., 2014b). Since total cooperation is assumed, this model effectively reduces to a single defender game in which the defender controls all resources. A recent extension, interdependent defense games (Chan et al., 2012), does consider an attacker who acts simultaneously with the defenders, rather than after observing the joint defense configuration, as in our model. Interdependent defense games have also been studied in the context of traffic infrastructure defense (Alderson et al., 2011). Two recent efforts studying multi-defender games explicitly model interdependence among targets through a probabilistic contagion process (Bachrach et al., 2013; Acemoglu et al., 2013). Like our paper, they consider attackers who observe the joint defense prior to making a decision, but each defender is restricted to secure a single node, and strategy space is assumed to be continuous. Vorobeychik et al. (2011) is, to our knowledge, the only other attempt to study strategic settings related to security in which each player's decision space is combinatorial. However, this work does not consider a strategic attacker.

#### 3. Multidefender Models

Our modeling effort proceeds in four steps, each generalizing the previous. As we see below, each generalization step reveals new and surprising insights about the multi-defender security setting, allowing us to appreciate the fundamental incentive forces. The first three models deal with homogeneous, independent targets and will be analyzed exactly using Nash equilibrium and price of anarchy (PoA) analysis. Our final model introduces interdependencies, and will be analyzed using computational methods.

#### 3.1 The Baseline Model

We start with a model which most reflects the related literature: in particular, this model involves n defenders and a single attacker, with each defender engaged in protecting a single target. Each target has the same value to the defender v > 0. We suppose that the defender has two discrete choices: to protect the target, or not. In addition, the defender can randomly choose among these; our focus is on these coverage probabilities (i.e., the probability of protecting, or covering, the target), which we denote by  $q_i$  for a given defender i. The attacker is strategic, can observe the defenders' coverage probabilities, and chooses a target that maximizes the damage. We assume that attacker is indifferent among the targets, and attacks the target with the lowest coverage probability, breaking ties uniformly at random. In a given scenario, for all defenders, the attacker's strategy is a vector of probabilities  $P = \langle p_1, p_2, ..., p_n \rangle$ , where  $p_i$  is the probability of attacking the target protected by defender i, with  $\sum_{i=1}^n p_i = 1$ .

We assume that if the attacker chooses to attack a target corresponding to defender i and defender i chooses to protect the target, then the utility of both is 0, and if the

attacker attacks the target but it is not protected, then the utility of the defender is -v while the attacker's utility is v. If a defender chooses to cover a target, it will incur a cost c>0. Additionally, we assume that targets are independent, i.e., if defender i is successfully attacked, all other defenders  $j\neq i$  receive 0 utility. We can thus define the expected utility of a defender i as

$$u_i = p_i u_i^a + (1 - p_i) u_i^u,$$

where  $u_i^a$  is the utility of i if it is attacked, and  $u_i^a$  is the utility of i if it is not attacked. By the assumptions above,

$$u_i^a = -(1 - q_i)v - q_i c = -v + q_i(v - c)$$
  
 $u_i^u = -q_i c.$ 

#### 3.2 The Multi-Target Model

Our key conceptual departure from related work is in allowing each defender to protect multiple targets, aligning it better with practical security domains. Specifically, suppose that there are n defenders, each protecting  $k \geq 1$  targets. Then the strategy of defender i will be a vector  $\langle q_{i1}, q_{i2}, ... q_{ik} \rangle$ . The strategy profile of the attacker can then be described as a matrix of probabilities,

$$\begin{pmatrix} p_{11} & p_{12} & \dots & p_{1k} \\ p_{21} & p_{22} & \dots & p_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nk} \end{pmatrix}$$

in which  $\sum_{i=1}^{n} \sum_{j=1}^{k} p_{ij} = 1$  and  $p_{ij} \geq 0$  for each i and j. The expected utility of a defender i in this model is

$$u_i = \sum_{i=1}^{k} p_{ij} u_{ij}^a + (1 - p_{ij}) u_{ij}^u,$$

where  $u_{ij}^a$  is the utility of target j to defender i if it is attacked, and  $u_{ij}^a$  is the utility of target j to i if it is not attacked. Using the notation introduced earlier, we have

$$u_{ij}^{a} = -(1 - q_{ij})v - q_{ij}c = -v + q_{ij}(v - c)$$
  
 $u_{ij}^{u} = -q_{ij}c.$ 

#### 3.3 The General Model

Generalizing further, we assume that if the attacker chooses to attack a target protected by defender i and the defender chooses to protect the target, then the utility of the target to defender i is  $U^c$ , and if the attacker attacks the target but it is not protected, then the utility of the target to the defender is  $U^u$ . It is reasonable to assume that  $U^c \geq U^u$ . If the target of defender i is not attacked, then we assume that the utility of the target for defender i is  $\Omega \geq U^c$ . Other assumptions are the same as those in the multi-target model. In the general model, therefore,

$$u_{ij}^{a} = q_{ij}U^{c} + (1 - q_{ij})U^{u} - q_{ij}c = (U^{c} - U^{u} - c)q_{ij} + U^{u}$$
$$u_{ij}^{u} = \Omega - q_{ij}c.$$

## 3.4 The Interdependent Model

The previous three models featured three important restrictions: first, that target values are homogeneous, second, that targets are independent, and third, that defenders protect the same number of targets. We now relax these restrictions. Suppose that a defender i can choose from a finite set O of security configurations for each target  $j \in T_i$ , where  $T_i$  is the set of targets under i's direct influence. Let T be the set of all targets, that is,  $T = \bigcup_i T_i$ , with |T| = m. A configuration  $o \in O$  for target  $j \in T_i$  incurs a cost  $c_j^o$  to the defender i. If the attacker attacks a target  $j \in T$  while configuration o is in place, the expected value to a defender i is denoted by  $U_{i,j}^o$ , while the attacker's value is  $V_j^o$ . We assume in this model that each player's utility depends only on the target attacked and its security configuration (Kiekintveld et al., 2009; Letchford & Vorobeychik, 2012). We denote by  $q_{i,j}^o$  the probability that the defender i chooses o at target  $j \in T_i$ .

While the problem we study assumes that that the utility of any player for a given target depends only on its security configuration o, there is a rather natural way to model interdependencies while retaining this structure, proposed by Letchford and Vorobeychik (2012). Specifically, suppose that dependencies between targets are represented by a graph (T, E), with T the set of targets (nodes) as above, and E the set of edges (j, j'), where an edge from j to j' means that a successful attack on j may have impact on j'. Each target j has associated with it a value,  $v_{ij}$ , for the defender i, which is the loss to i if j is affected (e.g., compromised, broken). The security configuration determines the probability  $z_i^o(j)$  that target j is affected if the attacker attacks it directly and the defense configuration is o. We model the interdependencies between the nodes as independent cascade contagion (Kempe et al., 2003; Letchford & Vorobeychik, 2012). The contagion proceeds starting at an attacked node j, affecting its network neighbors j' each with probability  $r_{i,i'}$ , the contagion then spreads from the newly affected nodes j' to their neighbors, and so on. The contagion can only occur one time along any network edge, and once a node is affected it stays affected through the diffusion process. Each player's valuation for each target is then updated based on the probability of a failure cascading to one of the player's owned targets.

## 3.5 The Weakness of Strong Stackelberg Equilibrium

By far the most important solution concept in Stackelberg security games is a *Strong Stackelberg equilibrium (SSE)*. A SSE is characterized by an assumption that the attacker breaks ties in defender's favor. When there is a single defender, this is well defined, and quite reasonable when the defender can commit to a mixed strategy: a slight adjustment in the defense policy will force the attacker to strictly prefer the desired option, with little loss to the defender. As we now illustrate, however, SSE is fundamentally problematic in a multi-defender context, because the notion of "breaking ties in defender's favor" is no longer well defined in general, as we must specify *which* defender will receive the favor.

To see concretely what goes wrong, consider the example in Figure 1. In this example there are two defenders, one who defends the target on the left, while the other defends the target on the right. Both defenders value their respective targets at 1, and have no value for the counterpart's target. The cost of defending each target is  $0 < \delta \ll 1$ . Now, consider a strategy profile in which  $q_j = 0$  for both targets t, and let us focus on the best

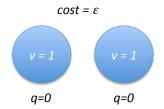


Figure 1: Example of a problem with a SSE in a multi-defender setting.

response of the first (left) defender. If this defender attempts to compute an SSE by fixing the strategy of the second player, he perceives his utility under the current strategy profile to be 1, since he would assume that the attacker breaks ties in his favor and, thus, attacks the defender on the right. By the same logic, the defender on the right will assume that the attacker will attack his counterpart, and perceive  $q_j = 0$  to be the best response. Since the attacker actually attacks one of them, the best response of the defender being attacked is to defend with a small probability, pushing the attacker towards the other target. What goes wrong here is that both players assume that the attacker attacks the other (breaks ties in their favor), which is inconsistent with the assumption that the attacker will certainly attack some target.<sup>1</sup>

#### 3.6 Solution Concepts

Since the classic (two-player) SSE solution concept used in Stackelberg security games does not conceptually extend to be an individual defender best-response problem in the multi-defender setting, we need to consider an alternative. One option is to compute an arbitrary subgame perfect equilibrium. However, we wish to impose a natural constraint on the solution concept that the attacker's best response be computed consistently for any joint defense policy, just as it is in a SSE (in other words, we wish to fix a tie-breaking rule). One natural tie-breaking rule is that the attacker chooses a target uniformly at random from the set of all best responses. We call the corresponding solution concept (which is a refinement of the subgame perfect equilibrium of our game) the Average-case Stackelberg Equilibrium (ASE). The crucial property of this solution concept that we desire is that the attacker's behavior presumed by a defender's best response problem is independent of that defender's identity, a property that SSE violates. As we demonstrate below, ASE is not guaranteed to exist, in which case we focus on  $\epsilon$ -equilibria, in which no defender gains more than  $\epsilon$  by deviating; in particular, we will consider  $\epsilon$ -equilibria with the smallest attainable  $\epsilon$ .

To measure how the efficiency of the game degrades due to selfish behavior of the defenders, we consider *Utilitarian Social Welfare* and  $(\epsilon)$ -Price of Anarchy in our paper. *Utilitarian Social Welfare* is the sum of all defenders' payoffs. For the smallest attainable

<sup>1.</sup> The problem we observe is similar to the issue of inconsistent conjectures the leaders in a multi-leader Stackelberg game could have about follower behavior noted by Kulkarni and Shanbhag (Kulkarni & Shanbhag, 2014). The solution we propose below—ASE—has the effect of imposing a shared constraint, the idea introduced by Kulkarni and Shanbhag generically. We note, however, that our concern here is not merely the fact that inconsistent conjectures lead to disequilibrium; rather, they could lead to nonsensical equilibria!

 $\epsilon$ , we define  $\epsilon$ -Price of Anarchy ( $\epsilon$ -PoA) as follows:

$$\epsilon - PoA = \frac{SW_O}{SW_E}$$

where  $SW_O$  is the optimal (utilitarian) social welfare that can be obtained (i.e., if there was a single defender), and  $SW_E$  is the worst-case (utilitarian) social welfare in  $\epsilon$ -equilibrium. An underlying assumption of this definition is that the value of  $SW_O$  and  $SW_E$  are both positive. If they are both negative, then  $\epsilon$ -PoA will be the reciprocal of above equation. Note that the ordinary Price of Anarchy is a special case of  $\epsilon$ -Price of Anarchy with  $\epsilon = 0$ .

## 4. Equilibrium Analysis of Independent Multi-Defender Security Games

In this section, we consider scenarios in which the values of the targets are *independent* and *homogeneous* among the defenders. Our equilibrium and Price of Anarchy analysis will show that a Nash equilibrium among defenders in the two-stage game model (equivalently, ASE) need not always exist, even when the defenders utilize randomized strategies (i.e., probability distributions over target protection levels). For cases when there is no Nash equilibrium, we make use of approximate Nash (ASE) equilibrium and the associated ( $\epsilon$ )-Price of Anarchy.

#### 4.1 The Baseline Model

Our first result presents necessary and sufficient conditions for the existence of a Nash equilibrium among defenders in the baseline model, and characterizes it when it does exist.

**Theorem 1.** In the Baseline model, Nash equilibrium exists if and only if  $v \ge c$ . In this equilibrium all targets are protected with probability 1.

*Proof.* Firstly, we claim that Nash equilibrium among defenders can appear only if all targets have the same coverage probability q. Otherwise, some defender j who has probability 0 of being attacked has the incentive to decrease her  $q_j$ . To find the Nash equilibria, we need only consider strategy profiles in which all targets have the same coverage probability.

When all defenders use the same coverage probability q, each defender's expected utility is

$$u = \frac{(v - cn)q - v}{n}.$$

If q < 1, some defender i could increase q to  $q + \delta$ , where  $\delta$  is a small positive real number, to avoid being attacked, and receive utility  $u' = -(q + \delta)c$ , so that

$$u' - u = \frac{v(1-q) - nc\delta}{n}.$$

As  $\delta$  can be arbitrarily small, u'-u>0 when q<1. Consequently, when q<1, a defender always has an incentive to deviate, which implies that the only possible Nash equilibrium can be for all players to play q=1.

When all defenders use coverage probability q = 1, each earns an expected utility of

$$u = -c$$
.

If a defender *i* decreases her coverage probability to q' < 1, then she will be attacked with probability 1, and receive expected utility u' = -v + q'(v - c), so that

$$u' - u = (v - c)(q' - 1).$$

If  $v \ge c$ , then  $u' - u \le 0$ , and q = 1 is indeed a Nash equilibrium. If v < c, however, u' - u > 0, which implies that a Nash equilibrium does not exist.

Thus, if a Nash equilibrium does exist, it is unique, with all defenders always protecting their targets. But what if the equilibrium does not exist? Next, we characterize the (unique)  $\epsilon$ -equilibrium with the minimal  $\epsilon$  that arises in such a case. We will use this approximate equilibrium strategy profile as a *prediction* of the defenders' strategies.

**Theorem 2.** In the Baseline model, if v < c, the optimal  $\epsilon$ -equilibrium is for all defenders to cover their target with probability  $\frac{v}{c}$ . The corresponding  $\epsilon$  is  $\frac{v(c-v)}{cn}$ .

*Proof.* We firstly consider strategy profiles in which all targets have the same probability q of being protected. Then each defender's expected utility is

$$u = \frac{(v - cn)q - v}{n}.$$

Suppose  $0 \le q < 1$ . If a defender *i* slightly increases *q* to  $q + \delta_1$ , she could receive a utility  $u' = -(q + \delta_1)c$ , with

$$u' - u = \frac{v(1-q) - nc\delta_1}{n} < \frac{v(1-q)}{n}.$$

Suppose  $0 < q \le 1$ . If a defender *i* slightly decreases *q* to  $q - \delta_2$ , she could receive utility  $u'' = -v + (q - \delta_2)(v - c)$ , with

$$u'' - u = \frac{v(1-q)(1-n) + \delta_2 n(c-v)}{n}.$$

Since  $\delta_2 \leq q$ ,

$$u'' - u \le \frac{v(1-q)(1-n) + qn(c-v)}{n} = \frac{v(1-q)}{n} + (qc - v).$$

Let  $d_1 = \frac{v(1-q)}{n}$ ,  $d_2 = \frac{v(1-q)}{n} + (qc - v)$ . If q = 0, a defender could deviate to increase utility by at most  $\frac{v}{n}$ . If q = 1, a defender could deviate to increase utility by at most (c - v). When  $0 < q \le \frac{v}{c}$ , we have that  $d_2 \le d_1$ , and it is  $d_1$ -equilibrium. When  $\frac{v}{c} < q < 1$ , we have that  $d_2 > d_1$ , and it is  $d_2$ -equilibrium.

Putting everything together, there is an  $\epsilon$ -equilibrium with

$$\epsilon = \begin{cases} \frac{v(1-q)}{n}, & \text{if } 0 \le q \le \frac{v}{c}; \\ \frac{v(1-q)}{n} + (qc - v), & \text{if } \frac{v}{c} < q \le 1. \end{cases}$$

Moreover,  $q = \frac{v}{c}$  gives us the unique minimal  $\epsilon = \frac{v(c-v)}{cn}$  when all defenders use the same coverage probabilities.

We now claim that the  $\frac{v(c-v)}{cn}$ -equilibrium could only exist when all defenders play an identical coverage probability. Suppose defenders use different coverage probabilities. Then there are  $\alpha$  defenders for  $1 \leq \alpha < n$  who have the same minimal probability q' of protecting their targets. The expected utility for each defender among these  $\alpha$  defenders is:

$$u_e = \frac{(v - c\alpha)q' - v}{\alpha}.$$

When  $\frac{v}{c} < q' \le 1$ , some defender *i* among these  $\alpha$  defenders could decrease her probability to 0 to get a utility of  $u_1 = -v$ , with

$$u_1 - u_e = \frac{v(1 - q')}{\alpha} + (q'c - v) > \frac{v(1 - q')}{n} + (q'c - v) > \frac{v(c - v)}{cn}.$$

When  $0 \le q' \le \frac{v}{c}$ , some defender *i* among these  $\alpha$  defenders could increase her probability to  $q' + \delta_3$  to get utility  $u_2 = -(q' + \delta_3)c$  with

$$u_2 - u_e = \frac{v(1 - q') - \alpha c \delta_3}{\alpha} > \frac{v(1 - q')}{n} \ge \frac{v(c - v)}{cn},$$

where the first inequality follows because  $\delta_3$  can be made arbitrarily small.

Armed with a complete characterization of predictions of strategic behavior among the defenders, we can now consider how this behavior related to socially optimal protection decisions. Since the solutions are unique, there is no distinction between the notions of price of anarchy and price of stability; we term the ratio of socially optimal welfare to welfare in equilibrium as the price of anarchy for convenience.

First, we characterize the socially optimal outcome.

**Theorem 3.** In the Baseline model, the optimal social welfare  $SW_O$  is

$$SW_O = \begin{cases} -cn, & \text{if } v \ge cn; \\ -v, & \text{if } v < cn. \end{cases}$$

Proof sketch. We firstly claim that we could get optimal social welfare only if all defenders use the same coverage probability q. If their coverage probabilities are different, and some defender j has probability 0 of being attacked, we could decrease  $q_j$  to improve social welfare. Therefore we need only to consider identical coverage probabilities q in determining optimal social welfare. Welfare, as a function of symmetric coverage q is

$$SW(q) = -v + q(v - c) + (n - 1)(-c) = (v - cn)q - v.$$

When  $v \geq cn$ , q = 1 is optimal, whereas q = 0 is optimal otherwise, giving the desired result.

From this result, it is already clear that defenders systematically over-invest in security, except when values of the targets are quite high. This stems from the fact that the attacker creates a *negative externality* of protection: if a defender protects his target with higher probability than others, the attacker will have an incentive to attack another defender.

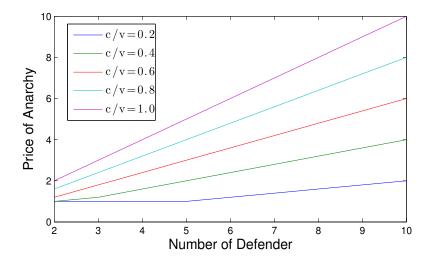


Figure 2: Price of Anarchy when  $v \geq c$ 

In such a case, we can expect a "dynamic" adjustment process with defenders increasing their security investment well beyond what is socially optimal. To see just how much the defenders lose in the process, we now characterize the price of anarchy of our game.

If  $v \geq c$ , social welfare in the unique equilibrium with q = 1 is

$$SW_E = -cn$$
.

The associated PoA is then

$$PoA = \begin{cases} 1, & \text{if } v \ge cn; \\ \frac{nc}{v}, & \text{if } c < v < cn. \end{cases}$$

Figure 2 shows the relationship among PoA, the number of defenders, and the ratio  $\frac{c}{v}$ . From the figure we can see that when number of defenders and  $\frac{c}{v}$  are small (e.g.  $n \le 5$  and  $\frac{c}{v} = 0.2$ ), the price of anarchy is close to 1. Otherwise, PoA is unbounded, growing linearly with n.

When v < c, there is no Nash equilibrium. However, the optimal  $\epsilon$ -equilibrium features all defenders with the same coverage probability  $\frac{v}{c}$  for their targets. The corresponding Social Welfare is

$$SW_E = (v - cn)\frac{v}{c} - v,$$

and the associated  $\frac{v(c-v)}{cn}$ -PoA is  $\frac{cn+c-v}{c}$ , which is, again, linear in n.

#### 4.2 The Multi-Target Model

Armed with observations from the model with a single target for each defender, we now extend the model to a case not as yet considered in the literature in a theoretical light: each defender protects a set of k targets. This gives rise to a combinatorial set of possible

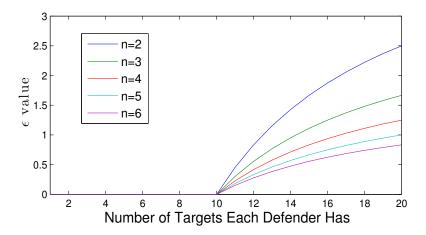


Figure 3:  $\epsilon$  value when v = 10, c = 1

decisions for each defender, so that even computing a best response is not necessarily easy. Remarkably, we are able to characterize equilibria and approximate equilibria in this setting as well. The proofs for this subsection are in the appendix.

Our first result is almost a mirror-image of the corresponding result in the baseline model: when a Nash equilibrium exists, all defenders protect all of their targets with probability 1.

**Theorem 4.** In the Multi-Target model, Nash equilibrium exists if and only if  $v \ge kc$ . In this equilibrium all targets are protected with probability 1.

Next, we consider scenarios when v < kc, in which there is no Nash equilibrium. Our next result characterizes optimal (lowest- $\epsilon$ ) approximate Nash equilibria.

**Theorem 5.** In the Multi-Target model, if v < kc, then in the optimal  $\epsilon$ -equilibrium all targets are protected with probability  $\frac{v}{kc}$ . The corresponding  $\epsilon$  is  $\frac{v(kc-v)}{cnk}$ .

Thus, as n increases, the optimal approximate equilibrium approaches a Nash equilibrium. Figure 3 illustrates the relationship between  $\epsilon$  and the number of targets each defender protects when v=10 and c=1. In this figure,  $\epsilon=0$  when  $k\leq 10$ , which means that an exact Nash equilibrium exists;  $\epsilon$  increases with k when k>10, but at a decreasing rate, converging to  $\frac{v}{n}$  as  $k\to\infty$ .

Finally, we characterize socially optimal welfare, and, subsequently, put everything together in describing the price of anarchy.

**Theorem 6.** In the Multi-Target model, the optimal social welfare  $SW_O$  is

$$SW_O = \begin{cases} -cnk, & \text{if } v \ge cnk; \\ -v, & \text{if } v < cnk. \end{cases}$$

Thus, just as in the baseline case, the defenders will generally over-invest in security.

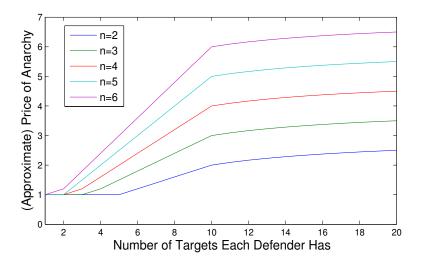


Figure 4: (Approximate) Price of Anarchy when  $\frac{c}{v} = 0.1$ 

If  $v \ge kc$ , there is a unique Nash equilibrium with all targets protected with probability 1. The corresponding social welfare is

$$SW_E = -cnk$$

Because it is the only Nash equilibrium, the *Price of Anarchy* is

$$PoA = \begin{cases} 1, & \text{if } v \ge cnk; \\ \frac{nkc}{v}, & \text{if } ck \le v < cnk. \end{cases}$$

If v < kc, there is no Nash equilibrium. The optimal approximate equilibrium features identical coverage probability of  $\frac{v}{kc}$  for all targets. The corresponding Social Welfare is

$$SW_E = (v - cnk)\frac{v}{kc} - v,$$

and the associated  $\frac{v(kc-v)}{cnk}$ -Price of Anarchy is  $n+1-\frac{v}{kc}$ . Clearly, in either case, and just as in the baseline model, the price of anarchy is unbounded, growing linearly with n.

We now consider how PoA changes as a function of k, i.e. the number of targets each defender has. When  $k \leq \frac{v}{cn}$ , a Nash equilibrium exists and the PoA is 1; when  $\frac{v}{cn} < k \leq \frac{v}{c}$ , PoA increases linearly in k with slope  $\frac{nc}{v}$ . However, when  $k > \frac{v}{c}$ , a Nash equilibrium does not exist and the approximate PoA is  $n+1-\frac{v}{kc}$ , which increases very slowly with k, and is bounded by n+1 when  $k \to \infty$ . Figure 4 illustrates the relationship between (approximate) Price of Anarchy and k for n=2. When k is very small, PoA = 1. For intermediate k, PoA increases linearly, and when k is sufficiently large, Nash equilibrium no longer exists, and  $\epsilon$ -PoA increases quite slowly, converging to 3 when  $k \to \infty$ .

#### 4.3 The General Model

Both the baseline and the multi-target models made rather strong assumptions about the structure of the utility functions of the players. In the general model, we relax these assumptions, allowing for arbitrary utilities for the players when the target is attacked or not, and when it is protected or not (when attacked). Quite surprisingly, our findings here are *qualitatively different*: the special case of the baseline and multi-target models turns out to be an exception, rather than the rule when more general models are considered.

Just as before, we start by characterizing Nash and approximate Nash equilibria.

**Theorem 7.** In the General model, Nash equilibrium exists if and only if  $U^c - U^u \ge kc - \frac{(n-1)(\Omega - U^c)}{n}$ . In this equilibrium all targets are protected with probability 1.

*Proof.* We firstly claim that Nash equilibrium can appear only if coverage probabilities of all of targets  $t_{ij}$  are identical. Otherwise, there will be a target  $t_{ik}$  which has the probability 0 of being attacked, and the defender i has an incentive to decrease  $q_{ik}$ . To determine a Nash equilibrium, we therefore need only consider scenarios in which all targets have the same coverage probability.

When all targets have the same coverage probability q to be protected, the utility of each defender is

$$u = \frac{(U^c - U^u - nkc)q + U^u + (nk - 1)\Omega}{n}.$$

If q < 1, then some defender i could increase q to  $q + \delta$  for all of her targets to ensure none of them are attacked, and obtain utility of  $u' = k\Omega - k(q + \delta)c$ , so that

$$u' - u = \frac{(U^c - U^u)(1 - q) + (\Omega - U^c) - nkc\delta}{n}.$$

As  $U^c \geq U^u$ ,  $\Omega \geq U^c$ , and  $\delta$  can be arbitrarily small, u' - u > 0 when q < 1, which means that this cannot be a Nash equilibrium. Thus, the only possible equilibrium can be  $q_{ij} = 1$  for all targets  $t_{ij}$ .

When all targets have the same coverage probability q=1, each defender's utility is

$$u = \frac{U^c - nkc + (nk - 1)\Omega}{n}.$$

We claim that if a defender i has an incentive to deviate, it is optimal for this defender to use the same coverage probability for all her targets. Otherwise, for some target  $t_{ik}$  which has probability 0 of being attacked, she could decrease  $q'_{ik}$  to obtain higher utility. If probabilities of targets protected by defender i are all q' ( $0 \le q' < 1$ ), then her expected utility is  $u' = (U^c - U^u - c)q' + U^u + (k-1)(\Omega - q'c)$ , and

$$u' - u = (U^c - U^u - kc)(q' - 1) + \frac{(n-1)(U^c - \Omega)}{n}.$$

We therefore have two cases:

1) If  $U^c - U^u \ge kc$ , then  $u' - u \le 0$ , and q = 1 for all targets is a Nash equilibrium.

2) If  $U^c - U^u < kc$ , the maximal value of u' - u corresponds to q' = 0:

$$\max_{0 \le q' < 1} u' - u = -(U^c - U^u - kc) - \frac{(n-1)(\Omega - U^c)}{n}.$$

If  $kc - \frac{(n-1)(\Omega - U^c)}{n} \le U^c - U^u < kc$ ,  $u' - u \le 0$ , it is a Nash equilibrium; otherwise, it is not.

To sum up, a Nash equilibrium exists if and only if  $U^c - U^u \ge kc - \frac{(n-1)(\Omega - U^c)}{n}$ , and the equilibrium corresponds to all targets having probability 1 of being protected.

Next, we characterize the optimal approximate equilibrium when no Nash equilibrium exists.

**Theorem 8.** In the General model, in the optimal  $\epsilon$ -equilibrium all targets are protected with probability  $\frac{\Omega - U^u}{kc}$ . The corresponding  $\epsilon$  is  $\frac{(\Omega - U^u)(kc - U^c + U^u)}{cnk}$ .

*Proof.* When all targets have the same coverage probability q, the expected utility of each defender is

$$u = \frac{(U^c - U^u - nkc)q + U^u + (nk - 1)\Omega}{n}.$$

Suppose  $0 \le q < 1$ . If some defender i increases q to  $q + \delta_{ij}$  for target  $t_{ij}$ , then she would obtain utility  $u' = \sum_{i=1}^{k} \Omega - (q + \delta_{ij})c$ , and

$$u' - u = \frac{\Omega - (U^c - U^u)q - U^u}{n} - \sum_{j=1}^k \delta_{ij}c$$

$$\leq \frac{\Omega - (U^c - U^u)q - U^u}{n}.$$
(1)

Now we consider scenarios in which a defender i could obtain higher utility by decreasing protection probability. We claim that if a defender i has an incentive to deviate, it is optimal for this defender to use the same coverage probability for all her targets. Otherwise, for some target  $t_{ik}$  which has probability 0 of being attacked, she could decrease  $q'_{ik}$  to obtain higher utility. Thus, we need only consider cases in which a defender deviates by decreasing coverage probabilities for all her targets to  $q - \delta$ . Her utility will become  $u'' = (U^c - U^u - kc)(q - \delta) + U^u + (k - 1)\Omega$ . Since  $U^c - U^u < kc$ ,  $\delta = q$  (the maximal value of  $\delta$ ) maximizes u'' - u:

$$\max_{0 < \delta \le q} u'' - u = \frac{\Omega - (U^c - U^u)q - U^u}{nk} + kcq + U^u - \Omega.$$
 (2)

By comparing the value of equation (1) and equation (2), we get different values of  $\epsilon$  for  $\epsilon$ -equilibrium:

$$\epsilon = \begin{cases} \frac{\Omega - (U^c - U^u)q - U^u}{n}, & \text{if } 0 \le q \le \frac{\Omega - U^u}{kc}; \\ \frac{\Omega - (U^c - U^u)q - U^u}{n} + kcq + U^u - \Omega, & \text{if } \frac{\Omega - U^u}{kc} < q \le 1. \end{cases}$$

When  $q = \frac{\Omega - U^u}{kc}$ , we get the minimal  $\epsilon = \frac{(\Omega - U^u)(kc - U^c + U^u)}{cnk}$ .

We claim that the  $\frac{(\Omega - U^u)(kc - U^c + U^u)}{cnk}$ -equilibrium can appear only if all targets have the same coverage probability q. We prove this by contradiction. Suppose that targets have different coverage probabilities. This gives rise to two cases: 1) Each defender uses an identical coverage probability for each target she owns (these may differ between defenders); and 2) Some defender has different coverage probabilities for her targets. In case 1), there exist  $\beta$  defenders  $(1 \le \beta < n)$  who have the same minimal coverage probability q'. The expected utility for each defender among these  $\beta$  is

$$u = \frac{(U^c - U^u - k\beta c)q' + U^u + (k\beta - 1)\Omega}{\beta}.$$

When  $\frac{\Omega - U^u}{kc} < q' \le 1$ , some defender i among these  $\beta$  could decrease the coverage probability of all her targets to 0 and obtain the utility of  $u_1 = U^u + (k-1)\Omega$ , so that

$$u_{1} - u = \frac{\Omega - (U^{c} - U^{u})q' - U^{u}}{\beta} + kcq' + U^{u} - \Omega$$
$$> \frac{\Omega - (U^{c} - U^{u})q' - U^{u}}{n} + kcq' + U^{u} - \Omega.$$

When  $0 \le q' \le \frac{\Omega - U^u}{kc}$ , some defender i among these  $\beta$  can increase coverage probabilities of all her targets to  $q' + \delta_3$  to obtain utility of  $u_2 = k\Omega - k(q' + \delta_3)c$ , with

$$u_2 - u = \frac{\Omega - (U^c - U^u)q' - U^u - k\beta c\delta_3}{\beta}$$
$$> \frac{\Omega - (U^c - U^u)q' - U^u}{n},$$

where the inequality holds because  $\delta_3$  can be arbitrarily small. Thus, no profile in case 1) can be a  $\frac{(\Omega-U^u)(kc-U^c+U^u)}{cnk}$ -equilibrium. In case 2), any defender who has different coverage probabilities among her targets can always increase her payoff by decreasing the coverage probabilities of the targets with higher coverage to yield identical coverage for all targets. Consequently, no profile in case 2) can be a  $\frac{(\Omega-U^u)(kc-U^c+U^u)}{cnk}$ -equilibrium.

As the final step towards characterizing the Price of Anarchy, we derive optimal social welfare in this model.

**Theorem 9.** In the General model, the optimal social welfare  $SW_O$  is

$$SW_O = \begin{cases} U^c - nkc + (nk - 1)\Omega, & \text{if } U^c - U^u \ge nkc; \\ U^u + (n - 1)\Omega, & \text{if } U^c - U^u < nkc. \end{cases}$$

Proof sketch. We firstly claim that we could get optimal social welfare only if all targets have the same coverage probability q. Otherwise, some target  $t_{ij}$  has probability 0 of being attacked, and we can decrease  $q_{ij}$  to improve social welfare. Consequently, we need only to consider an optimal symmetric coverage probability q to maximize social welfare, which can be done in a manner similar to that for the baseline case.

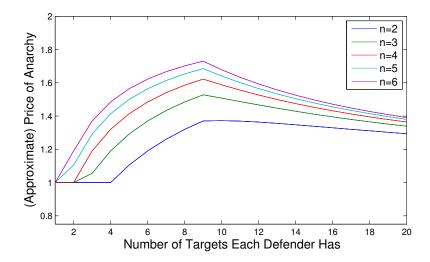


Figure 5: (Approximate) Price of Anarchy when  $c = 1, \Omega = -1, U^c = -2$  and  $U^u = -10$ 

If  $U^c - U^u \ge kc - \frac{(n-1)(\Omega - U^c)}{n}$ , the Nash equilibrium is unique, with all targets protected with probability 1. The corresponding social welfare is

$$SW_E = U^c - nkc + (nk - 1)\Omega.$$

So far we have not yet added any constraints to value of  $\Omega$ ,  $U^c$ , and  $U^u$  (except that  $\Omega \geq U^c \geq U^u$ ). In order to make *Price of Anarchy* well-defined, we need to add constraints that values of  $\Omega$ ,  $U^c$ , and  $U^u$  are all non-positive (just as in the previous two models) or all non-negative. To be consistent with previous models, we add constraints that  $U^c$ ,  $U^u$  and  $\Omega$  are all non-positive (little changes if all are non-negative).

In the case of a unique Nash equilibrium, the price of anarchy is

$$PoA = \begin{cases} 1, & \text{if } U^{c} - U^{u} \ge nkc; \\ \frac{U^{c} - U^{u} - nkc}{U^{u} + (nk - 1)\Omega} + 1, & \text{if } kc - \frac{(n - 1)(\Omega - U^{c})}{n} \le \\ U^{c} - U^{u} < nkc. \end{cases}$$

If  $U^c - U^u < kc - \frac{(n-1)(\Omega - U^c)}{n}$ , there is no Nash equilibrium. The Social Welfare in the optimal approximate equilibrium is

$$SW_E = (U^c - U^u - nkc)\frac{\Omega - U^u}{kc} + U^u + (nk - 1)\Omega,$$

and the  $\frac{(\Omega-U^u)(kc-U^c+U^u)}{cnk}$ -Price of Anarchy is  $\frac{(U^c-U^u-nkc)(\Omega-U^u)}{kcU^u+(nk-1)kc\Omega}+1$ .

We now analyze the relationship between  $(\epsilon$ -)PoA and the values of n and k. Here are the key differences from the Multi-Target Model. First we consider  $(\epsilon$ -)PoA as the function of n. If  $\Omega = 0$ , the result is same as that in the Multi-Target Model:  $(\epsilon$ -)PoA linearly increases in n, and is therefore unbounded. However, if  $\Omega \neq 0$ , while PoA and  $\epsilon$ -PoA are

increasing in n, as  $n \to \infty$ , they approach  $1 - \frac{c}{\Omega}$  and  $1 + \frac{U^u - \Omega}{k\Omega}$ , respectively. In other words, PoA (exact and approximate) is bounded by a constant, for a constant k!

Consider now approximate price of anarchy as a function of k. If  $\Omega=0$ , it is bounded by n+1. However, if  $\Omega \neq 0$ , when  $kc - \frac{(n-1)(\Omega - U^c)}{n} \leq U^c - U^u$ , it is an increasing function of k. When  $kc - \frac{(n-1)(\Omega - U^c)}{n} > U^c - U^u$ , it may at first increase or decrease in k, depending on the the values of the model parameters. However, when k is large enough, price of anarchy will invariably be decreasing in k, and as  $k \to \infty$ ,  $\epsilon$ -PoA  $\to 1$ . Figure 5 provides an example of the relationship between  $\epsilon$ -PoA and k. Observe that all the curves begin to decrease when k > 10, and they all approach 1 as  $k \to \infty$ . Thus, price of anarchy in the general model is only unbounded in the special case when  $\Omega = 0$ , whereas when  $\Omega \neq 0$ , price of anarchy is always bounded by a constant. This observation is particularly surprising and significant considering the fact that the baseline and simplified multi-target models are quite natural, and seemingly innocuous, restrictions of the general case.

## 5. Analysis of Interdependent Multi-defender Security Games

We now develop and analyze a computational framework for approximating Nash equilibria in *interdependent* multi-defender security games. A crucial step in computing (or approximating) a Nash equilibrium of a game is to consider the problem of computing a best response for an arbitrary player (in our case, defender, since the attacker's best response is straightforward). Next, we develop a novel mixed-integer linear programming formulation for computing ASE best response, and then propose a hightly effective heuristic method for approximating ASE in multi-defender games.

# 5.1 Computing Defender Best Response: A Mixed-Integer Linear Programming Formulation

While ASE seems a very natural alternative to SSE even in two-player security games, we are not aware of any proposals for computing it. Below, in equations 3-13, we present the first (to our knowledge) mixed-integer linear programming formulation for computing ASE which, in our case, would compute a best response for an arbitrary defender i when the strategies of all other players,  $q_{-i}$ , are fixed.

$$\max_{a,q_{i},s,u,v} u - \sum_{j \in T_{i}} \sum_{o \in O} c_{j}^{o} q_{i,j}^{o}$$
(3)

s.t.

$$0 \le q_{i,j}^o \le 1 \qquad \forall j \in T_i, \ \forall o \tag{4}$$

$$\sum_{o \in O} q_{i,j}^o = 1 \qquad \forall j \in T_i \tag{5}$$

$$a_j \in \{0, 1\} \tag{6}$$

$$\sum_{j \in T} a_j \ge 1 \tag{7}$$

$$0 \le v - \sum_{o} q_{i,j}^{o} V_j^{o} \le (1 - a_j) M \qquad \forall j \in T_i$$
 (8)

$$0 \le v - \sum_{o} q_{-i,j}^{o} V_{j}^{o} \le (1 - a_{j})M \qquad \forall j \in T_{-i}$$
 (9)

$$s_j = v - \sum_o q_{i,j}^o V_j^o \qquad \forall j \in T_i$$
 (10)

$$s_j = v - \sum_{o} q_{-i,j}^o V_j^o \qquad \forall j \in T_{-i}$$
 (11)

$$a_j + Ms_j \ge 1 \tag{12}$$

$$u = f(q, a), \tag{13}$$

where M is a very large number and

$$f(q, a) = \frac{\sum_{j \in T_i} a_j \sum_{o \in O} q_{i,j}^o U_j^o + \sum_{j \in T_{-i}} a_j \sum_{o \in O} q_{-i,j}^o U_j^o}{\sum_{j \in T} a_j}.$$

While constraint 13 is non-linear, we can linearize it using McCormick inequalities. Constraints 4 and 5 ensure that the defender's strategy is a valid probability distribution. Constraint 7 ensures that at least one target is chosen by the attacker. Constraints 8 and 9 compute the optimal attacker utility v; alone, they ensure that this utility corresponds to some attack target. Constraints 10 and 11 compute an auxiliary variable  $s_j$ , which is 0 if and only if attacking a target j yields an optimal utility to the attacker. These variables, together with constraints 12 and 8-9 ensure that the binary variable  $a_j = 1$  if and only if the attacker (weakly) prefers to attack target j; that is, these jointly compute the set of optimal attack targets. Finally, constraint 13 computes the expected utility to the defender if the attacker chooses one of his most preferred targets uniformly at random.

If M is infinite and numbers can be computed to arbitrary precision, the above formulation is correct. In practice, of course, numerical precision and stability are an issue, and they arise with this formulation. Consider constraints 10 and 11, which compute  $s_j$  for all targets j. These require that the expected value of the target exactly equal the optimal attacker utility computed in constraints 8-9; even a slight error will technically violate our requirement that  $s_j = 0$  at an optimal target. Moreover, even if the difference is, indeed, non-zero, from an attacker's perspective it seems intuitive that sufficiently small differences from optimal utility are ignored. We address these problems by adding a fixed small quantity  $\delta$  to the right-hand-side of constraints 8, 9, while subtracting it from the right-hand-side of constraint 12. Because the constraints are interrelated, we cannot simply choose an arbitrary  $0 < \delta < 1$ , but must ensure that  $\delta$  and  $\delta$  satisfy the constraint that  $\delta < 1 - \delta$ .

## 5.2 Approximating ASE

Previously, Vorobeychik and Wellman (2008) presented a convergent equilibrium approximation algorithm based on *simulated annealing (SA)* that would be applicable in our setting. They additionally showed in simulation that SA is actually outperformed by a simple heuristic based on *iterated best response (IBR)* dynamics. Here, we interpret IBR as a local search heuristic, with the property that if the starting point is a Nash equilibrium, IBR will never deviate from it (i.e., Nash equilibrium is a fixed point). Clearly, then, the choice of a starting point can be significant for the performance of IBR, making it natural to consider coupling

it with random restarts. Our main contribution in this section is to present evidence that IBR with random restarts is a highly effective equilibrium approximation approach in our setting (and outperforms several alternatives). This is both of broad significance, and of particular importance in our setting, as we use this algorithm for our analyses below.

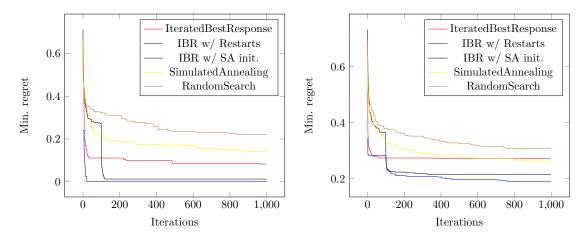


Figure 6: Comparison of algorithms. Left: |D|=2 and |T|=10. Right: |D|=5 and |T|=20.

We compare the following Nash equilibrium approximation algorithms executed for 1000 iterations: random search (RS), which simply generates 1000 strategy profiles randomly, computes the game theoretic regret of each, and chooses a profile with the smallest regret; simulated annealing (SA), with the temperature exponentially increasing with iterations; and iterated best response (IBR) with no restarts. We also include in the comparison two additional variations of IBR: the first uses SA for the first 100 iterations, and then switches to IBR for the remainder (starting with the best approximation produced by SA); the second is IBR with random restarts, which we term RIBR. RIBR includes initial corner cases that may be hard to converge to in a limited amount of time (i.e., all defenders not defending, all defenders defending completely). We execute our comparison on games with 2 players and 10 targets and games with 5 players and 20 targets. In all cases, targets are divided evenly among the players, and values over the targets are generated uniformly at random. The cost of defense is fixed at c=0.2, and the targets are assumed to be independent (but players may have values for targets under the control of other defenders). Figure 6 demonstrate that in both settings, RIBR outperforms other alternatives.

#### 5.3 Analysis of Multi-Defender Games on Synthetic Networks

For our first set of experiments, we use RIBR on 3 artificially generated networks, with 40 samples for each parameter variation. First, we will illustrate and compare the results of our interdependent multi-defender game on artificial networks. We use 3 commonly analyzed network structures: a grid, Erdős-Rényi networks, and preferential attachment networks. In all of the generated networks, there are 64 nodes or targets. For the latter two, we use the Metis graph partitioning software to partition the nodes (targets) among defenders. This

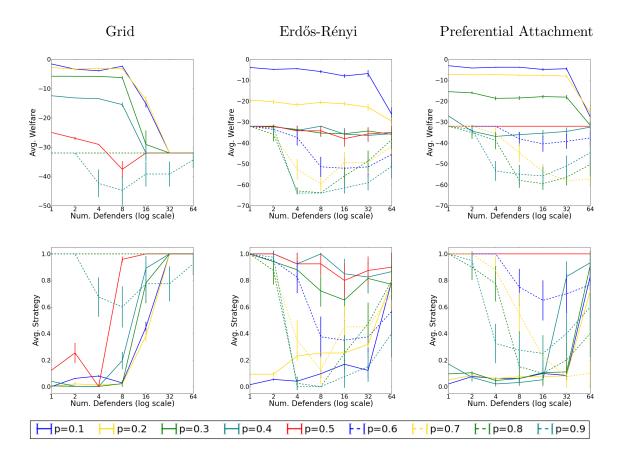


Figure 7: Approximate equilibrium security outcomes for varying cascade probability p, as a function of (log of) the number of players (level of decentralization) for the three synthetic networks. Top: Social welfare. Bottom: Average strategy (higher strategy corresponds to higher average probability of defense).

software partitions nodes to minimize connectivity among the targets belonging to different defenders, a property that we expect to common hold in real networks due to efficiency considerations.

We begin by considering average strategies, as well as social welfare, for the three different synthetic networks (grid, Erdős-Rényi, and preferential attachment), as a function of the number of players (degree of decentralization) and the cascade probability (interdependent risk). The results are shown in Figure 7. We do not show these as a function of defense cost as increasing defense cost roughly mirrors decreasing cascade probability p. The first rather stark observation is that network structure makes little difference when each node is controlled by a single player, but it makes a significant qualitative difference both for social welfare and actual strategies utilized by the players in all other cases.

Looking at the results in greater detail, let's consider first social welfare (Figure 7, top). First, when interdependent risk is low  $(0.1 \le p \le 0.3)$ , social welfare follows a relatively simple pattern: increasing decentralization makes initially almost no difference, until sufficiently

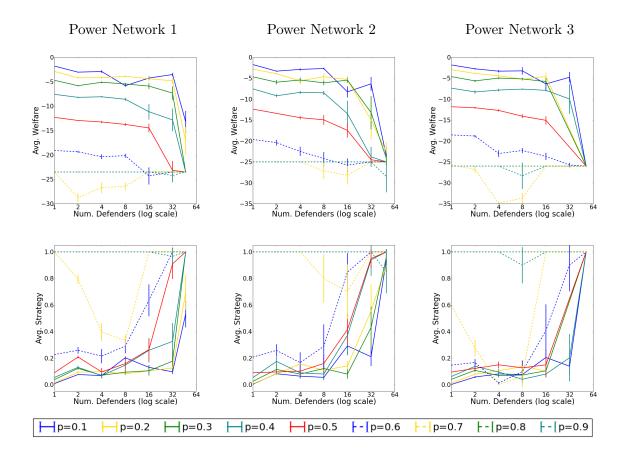


Figure 8: Approximate equilibrium security outcomes for varying cascade probability p, as a function of (log of) the number of players (level of decentralization) for the three real power networks. Top: Social welfare. Bottom: Average strategy (higher strategy corresponds to higher average probability of defense).

many players are involved, at which point social welfare falls rather dramatically; this pattern is roughly monotonic with increasing decentralization, with worst outcomes emerging when each player controls a single node, and mirrors previous findings (Vorobeychik et al., 2011). Both Erdős-Rényi and preferential attachment networks are less susceptible to the negative effects of decentralization in this case than the grid network, where the dropoff occurs with fewer players (less decentralization). This may be largely a consequence of the fact that network partitioning tools we use attempt to minimize interdependence among players—something that is likely to mirror reality—and far more opportunities for doing so exist in Erdős-Rényi and preferential attachment models.

When p is higher (greater interdependencies), the results exhibit an entirely new phenomenology. Across all three network models, for a sufficiently large p, the impact of decentralization is non-monotonic: an intermediate level of decentralization has the most detrimental impact on security, while a highly decentralized system becomes near-optimal!

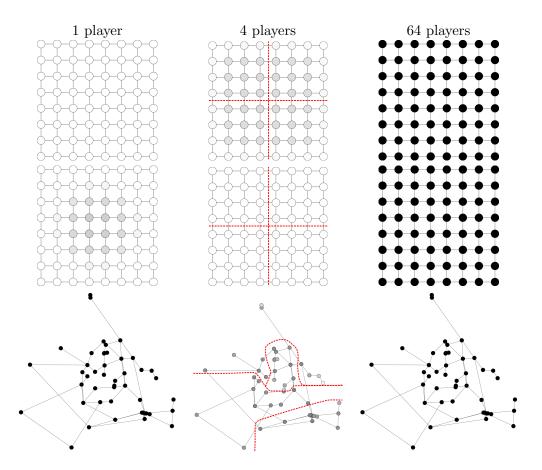


Figure 9: Strategic realizations for representative games on grid and distribution network topologies. Top: Grid, p = 0.1, middle: Grid, p = 0.4, bottom: Power Network, p = 0.7. Darker node colors indicate higher probability of defense (coverage); so white implies no coverage and black implies that the node is always covered (defended). Dotted red lines indicate the partition of nodes among players (except in the rightmost case, when each player controls a single node).

Investigating actual (average) strategic decisions by the players yields deeper insights into the findings above. When interdependencies are weak, optimal decision is to invest relatively little in security, in any generative model. Increasing decentralization, therefore, gives rise primarily to over-investment, mirroring our analytical results for the limiting case when targets were independent, although the tendency to over-invest is quite weak until the network is extremely decentralized, except in the grid network. When p is high, on the other hand, the predominant phenomenon is underinvestment. This in itself is not surprising: after all, a high level of interdependencies should imply that positive externalities of security should be dominant. What is surprising is, again, non-monotonicity in the level of decentralization: when decentralization is moderate, underinvestment can be quite dramatic. On the other hand, a high level of decentralization often appears to dull this effect, and the level of investment in security becomes much closer to optimal.

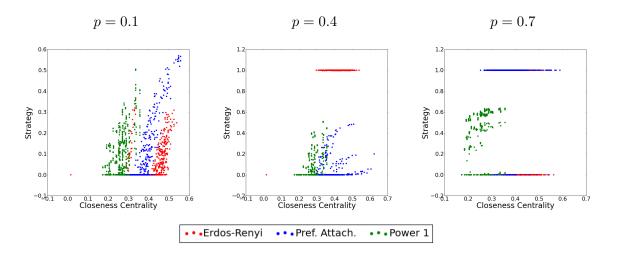


Figure 10: Node security and network (closeness) centrality; each defender controls 4 nodes.

## 5.4 Results on power grid networks

The grid network studied above is arguably the most "artificial", in the sense that both Erdős-Rényi and preferential attachment models were developed in part to resemble real networks (this is particularly true of the latter, which aims to replicate the scale-free properties of observed networks). Surprisingly, however, the approximate equilibrium results applied to three snippets of actual power networks most resemble the phenomena observed for the grid, as can be seen in Figure 8. In particular, just as in the grid above, over-investment in security appears to dominate, even at relatively high levels of interdependence, but only when decentralization is significant, while most levels of decentralization are relatively robustly near-optimal (these are, in fact, more robust to decentralization than the grid network above).

To dig somewhat deeper into the rather complex phenomenology we have observed, Figure 9 shows several examples of actual strategy realizations. First, consider the top series of plots for the grid with cascade probability p = 0.1. As previously described, we can clearly see that the optimal security configuration involves no security investment (leftmost grid), whereas an increasing level of decentralization gives rise to increased security investment, culminating, ultimately, with full protection in the extreme level of decentralization. The contrast between the two extremes offers some guidance: even though optimal global configuration involves no security, when each player controls (and cares about) only a single node, the best response of an attacked node is to defend it just enough to force the attacker to attack another; for example, slightly more than the next weakest node. Iterating on this idea, strategies "cascade" to full defense. When the player controls more than one node, however, there is suddenly strategic tension: higher security on one node may well push the attacker to attack another node under this player's control. Positive externalities become more significant as well: pushing the attacker to attack another node "nearby" is likely to gain little when cascade probabilities are high and multiple nodes owned by the defender could be affected. For sufficiently high cascade probabilities, and sufficiently low number of players, such positive network effects can actually sway players to under-invest in security, as we can see both in the middle and last rows of Figure 9 (the 4-player case). Here, strategic complementarities make security investment not worthwhile in equilibrium: the nodes that need to be defended are relatively central, and cut across different players (i.e., the critical central nodes create a kind of "buffer" between defenders). This behavior diminishes as decentralization increases.

#### 5.5 Security and Network Centrality

Finally, we consider the relationship between network centrality and strategic choice of a node. The results, shown in Figure 10, plot each node's closeness centrality and corresponding security across 10 instances for each network. When the interdependence of networks is low, security investment appears to correlate rather strongly with closeness in synthetic networks: in other words, nodes more central in the network invest more in security. A similar relationship is seen with node degrees. This correlation, however, largely disappears with greater interdependence, likely because the difference between being one-hop vs. two-hops away from an attacked node (i.e., being a low-connected neighbor of a high-connected node) becomes considerably less in such a case.

For the power networks, the correlation between centrality and strategy remains even with higher cascade probabilities. This relationship can be seen by comparing the strategy profile in the second column of Figure 9 and the closeness plot for p=0.7 for the power network. The structure of the power network has several small "chains" that have low centrality. The node that connects the chain to the rest of the network (higher centrality) incurs most of the defense cost, since a failure cascade would have to pass through this node to reach the rest of the chain. As the amount of interdependence increases (column 3 in Figure 9), these chains become partitioned between defenders, weakening the relationship between centrality and strategy.

#### 6. Conclusion

In this work, we have extended the current state of Stackelberg security games to include multiple defenders in non-cooperative scenarios with independent and interdependent targets.

For the independent case, we provided complete characterizations of Nash and approximate equilibria, socially optimal solutions, and price of anarchy (PoA) for three models of varying generality. Our analysis showed that defenders generally over protect the targets, but different modelling assumptions give rise to qualitatively different outcomes: a simpler model gives rise to an unbounded PoA, whereas a more general model sees PoA converge to a constant when the number of defenders increases.

For the interdependent case, we developed a novel computation framework to overcome the difficulties of providing a concise formal analysis of such a complex model. Our simulations characterize a broad space of strategic predicaments, varying cascade probability, network structure, and system decentralization. In contrast to the independent models, our results show differing behavior in terms of security investment dependent on the strength and structure of interdependencies. One of our most stark findings is the non-monotonicity of welfare and strategic choices as a function of the number of players: in a number of cases, higher levels of decentralization become near-optimal, even while intermediate decentralization leads to very poor outcomes. As security decisions are almost universally decentralized, and often highly interdependent, our findings enable a deeper understanding of practical security considerations, highlighting the importance of *both*, over- and underinvestment in security, and the dependence of each on network structure, the magnitude of network externalities, and the level of decentralization. Finally, we have shown how security behavior in our model on real-world power networks relates to those in synthetic networks, highlighting similar behaviors with grid networks, and unveiling structural differences with Erdős-Rényi and preferential attachment networks using the relationship between strategy and centrality.

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## Appendix

**Theorem 4.** In the Multi-Target model, Nash equilibrium exists if and only if  $v \ge kc$ . In this equilibrium all targets are protected with probability 1.

*Proof.* We firstly claim that a Nash Equilibrium must have identical coverage probabilities for all targets  $t_{ij}$ . Otherwise, there will be a target  $t_{ik}$  which has the probability 0 of being attacked, and defender i has an incentive to decrease  $q_{ik}$ .

When all targets have the same coverage probability q, the expected utility of each defender is

$$u = \frac{(v - cnk)q - v}{n}.$$

If q < 1, then some defender i can increase q to  $q + \delta$  for all of her targets to make sure none are attacked, obtaining utility  $u' = -k(q + \delta)c$ , with

$$u' - u = \frac{v(1-q) - nkc\delta}{n}.$$

As  $\delta$  can be arbitrarily small, u'-u>0 when q<1, and it cannot be a Nash equilibrium. When all targets have the same coverage probability q=1, the expected utility of each defender is u=-kc. If defender i want to deviate, then one of her targets will be attacked. Suppose this target is  $t_{ir}$ , with  $q_{ir}<1$ . Then her expected utility is

$$u' = -v + q_{ir}(v - c) + \sum_{j=1, j \neq r}^{k} (-q_{ij}c),$$

and

$$u' - u = (v - c)(q_{ir} - 1) + \sum_{j=1, j \neq r}^{k} (-q_{ij}c) + (k - 1)c.$$

We now have two cases:

- 1) If v < c, then  $(v c)(q_{ir} 1) > 0$  and  $\sum_{j=1, j \neq r}^{k} (-q_{ij}c) + (k-1)c \ge 0$ , and the defender has an incentive to deviate, and there is no Nash equilibrium.
- 2) If  $v \geq c$ , then, because  $q_{ij} \geq q_{ir}$  for all  $j \neq r$ ,

$$u' - u \le (v - c)(q_{ir} - 1) + (k - 1)c - (k - 1)q_{ir}c$$

and

$$u' - u \le (q_{ir} - 1)(v - kc).$$

In case 2, therefore,  $u' - u \le 0$  iff  $v \ge kc$ , which yields the desired result.

**Theorem 5.** In the Multi-Target model, if v < kc, then in the optimal  $\epsilon$ -equilibrium all targets are protected with probability  $\frac{v}{kc}$ . The corresponding  $\epsilon$  is  $\frac{v(kc-v)}{cnk}$ .

*Proof.* When all defenders have the same coverage probability q, the expected utility of each defender is

$$u = \frac{(v - cnk)q - v}{n}.$$

Suppose  $0 \le q < 1$ . If some defender i increases q to  $q + \delta_{ij}$  for target  $t_{ij}$ , then she would obtain a utility of  $u' = \sum_{j=1}^{k} -(q + \delta_{ij})c$ , and

$$u' - u = \frac{v(1-q)}{n} - \sum_{j=1}^{k} \delta_{ij}c < \frac{v(1-q)}{n}.$$

We claim that if a defender i has an incentive to deviate, it is optimal for this defender to use the same coverage probability for all her targets. Otherwise, for some target  $t_{ik}$  which has probability 0 of being attacked, she could decrease  $q'_{ik}$  to obtain higher utility. Therefore, we need only consider cases when a defender deviates by decreasing probabilities of all her targets to  $q - \delta$ . Then her utility is  $u'' = (v - kc)(q - \delta) - v$ , and

$$u'' - u = \frac{\delta n(kc - v) + v(n - 1)(q - 1)}{n}.$$

Since v < kc when  $\delta = q$  (the maximal value of  $\delta$ ), the maximal value of u'' - u is

$$\max_{0 < \delta \le q} u'' - u = \frac{v(1-q)}{n} + kcq - v.$$

Let  $d_1 = \frac{v(1-q)}{n}$ ,  $d_2 = \frac{v(1-q)}{n} + kcq - v$ . Then,

$$d_1 - d_2 = -kcq + v$$

When  $q \leq \frac{v}{kc}$ ,  $d_1 \geq d_2$ , and it is a  $\frac{v(1-q)}{n}$ -Nash Equilibrium; when  $q > \frac{v}{kc}$ ,  $d_1 < d_2$ , and it is a  $(\frac{v(1-q)}{n} + kcq - v)$ -Nash Equilibrium. To sum up, for  $\epsilon$ -Nash Equilibrium,

$$\epsilon = \begin{cases} \frac{v(1-q)}{n}, & \text{if } 0 \le q \le \frac{v}{kc}; \\ \frac{v(1-q)}{n} + kcq - v, & \text{if } \frac{v}{kc} < q \le 1. \end{cases}$$

When  $q = \frac{v}{kc}$ , we obtain the minimal  $\epsilon = \frac{v(kc-v)}{cnk}$ . We claim that the  $\frac{v(kc-v)}{cnk}$ -Nash Equilibrium can appear only if all targets have the same coverage probability q. Suppose targets have different coverage probabilities. This implies two possibilities: 1) Each defender uses identical coverage probabilities for her targets, and differences exist only between defenders; 2) Some defender uses diverse coverage probabilities among her own targets.

We first consider case 1), in which targets may have different probabilities of bing protected, but each defender has the same probability of protecting her own targets. In this case there exist  $\beta$  defenders  $(1 \le \beta \le n)$  who have the same minimal coverage probability q'. The expected utility for each defender among these  $\beta$  defenders is:

$$u_e = \frac{(v - kc\beta)q' - v}{\beta}.$$

When  $\frac{v}{kc} < q' \le 1$ , some defender i among these  $\alpha$  defenders can decrease probability of all her targets to 0 to obtain a utility of  $u_1 = -v$ , with

$$u_1 - u_e = \frac{v(1 - q')}{\beta} + (kcq' - v) > \frac{v(1 - q')}{m} + (kcq' - v).$$

When  $0 \le q' \le \frac{v}{c}$ , some defender i among these  $\beta$  defenders can = increase probability of all her targets to  $q' + \delta_3$  to obtain a utility of  $u_2 = -k(q' + \delta_3)c$ , with

$$u_2 - u_e = \frac{v(1 - q') - kc\beta\delta_3}{\beta} > \frac{v(1 - q')}{n},$$

where inequation holds because  $\delta_3$  can be arbitrarily small. Consequently, case 1 cannot have a  $\frac{v(kc-v)}{cnk}$ -Nash Equilibrium.

Now we consider case 2), in which there exists a defender who uses different coverage probabilities among her targets. As some of her targets have probability 0 of being attacked, she could increase her payoff by decreasing probabilities of all of these targets to be as small as her target with the lowest coverage probability. It means that each coverage profile in case 2 can always be transformed to a coverage profile in case 1 in a way that improves a defender's utility without hurting anyone else. This means that case 2 also cannot have a  $\frac{v(kc-v)}{cnk}$ -Nash Equilibrium.

**Theorem 6.** In the Multi-Target model, the optimal social welfare  $SW_O$  is

$$SW_O = \begin{cases} -cnk, & if \ v \ge cnk; \\ -v, & if \ v < cnk. \end{cases}$$

Proof sketch. We firstly claim that we could obtain optimal social welfare only if all targets have the same coverage probability q. Otherwise, some target  $t_{ij}$  has the probability 0 of being attacked, and we can decrease  $q_{ij}$  to improve social welfare. Consequently, we need only to consider an optimal identical coverage probability q to obtain optimal social welfare, which can be done in a way similar to the baseline case.